Arrangements of symmetric products of spaces

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Abstract

We study the combinatorics and topology of general arrangements of subspaces of the form $D + SP^{n-d}(X)$ in symmetric products $SP^n(X)$ where $D \in SP^d(X)$. Symmetric products $SP^m(X) := X^m/S_m$, also known as the spaces of effective "divisors" of order m, together with their companion spaces of divisors/particles, have been studied from many points of view in numerous papers, see [8] and [22] for the references. In this paper we approach them from the point of view of geometric combinatorics. Using the topological technique of *diagrams of spaces* along the lines of [35] and [38], we calculate the homology of the union and the complement of these arrangements. As an application we include a computation of the homology of the *homotopy end space* of the open manifold $SP^n(M_{g,k})$, where $M_{g,k}$ is a Riemann surface of genus g punctured at k points, a problem which was originally motivated by the study of commutative $(m+k, m)$-groups [33].

1 Arrangements of symmetric products

The study of homotopy types of arrangements of subspaces with an emphasis on the underlying combinatorial structure is a well establish part of geometric and topological combinatorics. Originally the focus was on the arrangements of linear or affine subspaces [7] [18] [27] [37]. Gradually other, more general arrangements of spaces were introduced and studied. Examples include arrangements of pseudolines/pseudospheres, in connection with the realizations of (oriented) matroids [6] [31], arrangements of projective and Grassmann varieties, partially motivated by a geometrization of the Stanley ring construction [16] [35], arrangements of classifying spaces BH for a family of subgroups of a given group, subspace arrangements over finite fields [5] etc. With the introduction into combinatorics of the technique of *diagrams of spaces* and the associated *homotopy colimits* [35] [38], it became apparent that arrangements of subspaces have much in common with other important and well studied objects like stratified spaces/discriminants and their geometric resolutions [34], toric varieties viewed as combinatorial objects associated to face latices of polytopes [10] [12] [35] etc. All this serves as a motivation for the study of general subspace arrangements carrying interesting combinatorial structure.

Symmetric products of spaces $SP^n(X)$ are classical mathematical objects [1] [2] [11] [13] [14] [15] [22] [28] which appear in different areas of mathematics and mathematical physics as orbit spaces, divisor spaces, particle spaces etc., see [8] for a leisurely introduction and a review of old and new applications. The case of 2-manifolds M is of particular interest since in this case $SP^n(M)$ is a manifold. Elements $D \in SP^d(M)$

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are called divisors of order $|D| = d$. In this paper we study arrangements in $SP^n(X)$ of the form

$$\mathcal{A} = \{D_i + SP^{n-|D_i|}(X)\}_{i=1}^k, \tag{1}$$

where the case of open or closed surfaces is of special interest.

Given an arrangement $\mathcal{A} = \{F_1, \ldots, F_k\}$ of subspaces in an ambient space V, the *union* or the *link* of $\mathcal{A}$ is $D(\mathcal{A}) := \bigcup_{i=1}^k F_i$ and the complement is $M(\mathcal{A}) := V \setminus D(\mathcal{A})$. In this paper we compute the homology of the link and, in the case of Riemann surfaces M_g, the homology of the complement of the arrangement (1). As an application we compute the homology of the "homotopy-end-space" of $SP^n(M_{g,k})$ where $M_{g,k} := M_g \setminus \{x_1, \ldots, x_k\}$ is the so called (g,k)-amoeba, see Figure 1, and discuss the connection of this computation with the problem of existence of commutative $(m+k,m)$-groups and the problem what information about the original surface M can be reconstructed from the symmetric product $SP^n(M)$, Section 2.3.

Note that if $g = 0$ and if $D_i = x_i$ are all distinct divisors of order 1, then the complement of the arrangement (1) is homeomorphic to the complement of a generic arrangement of $k-1$ hyperplanes in $\mathbb{C}^n$, studied by Hattori [19] [27]. So the results about $SP^n(M_{g,k})$ can be viewed as an extension of some classical results about complex hyperplane arrangements. On the technical side, we would like to emphasize the role of the relation of *proper* domination between simple diagrams, Definition 1.6 in Section 1.1, which often allows, as demonstrated in Section 1.3, the study of naturality properties for Goresky-MacPherson [16] and Ziegler–Živaljević [38] [35] type formulas. The same concept allows us to extend (Theorems 1.8 and 1.11) the classical Steenrod's theorem on the decompositions od symmetric products, to the case od diagrams of spaces.

1.1 Homology of the union $D(\mathcal{A}) = \bigcup \mathcal{A}$

We approach the computation of the homology $H_*(D(\mathcal{A}))$ by the method of *diagrams of spaces*. The references emphasizing applications of this technique in geometric combinatorics are [35] and [38]. The reader is referred to these papers for the notation and standard facts. Aside from standard tools like the *Projection Lemma* or the *Homotopy Lemma*, we make a special use of the idea of an "ample space" diagram outlined in Section 5.4., of [35].

Let $\mathcal{A} = \{F_n^i\}_{i=1}^k$ be an arrangement of subspaces in $SP^n(X)$ where $F_n^i := x_i + SP^{n-1}(X) \cong SP^{n-1}(X)$ and $\{x_i\}_{i=1}^k$ is a collection of distinct points in X. Let $P = P(\mathcal{A})$ be the intersection poset of $\mathcal{A}$. By definition [27] [35] [38], P has an element for each non-empty intersection $F_n^I = F_n^{i_0} \cap F_n^{i_1} \cap \ldots \cap F_n^{i_p}$ where $I = \{i_0, i_1, \ldots, i_p\} \subset [k] := \{1, \ldots, k\}$. If $n \geq k$ then P is isomorphic to the power set $\mathcal{P}'[k] = \mathcal{P}[k] \setminus \{\emptyset\}$ or alternatively, the face poset of an abstract simplex Σ with vertices $\{1, \ldots, k\}$. If $n \leq k$ then P is isomorphic to the poset $\mathcal{P}'_{\leq n}[k]$ of all non-empty subsets I of cardinality at most n or alternatively the $(n-1)$-skeleton Σ^{n-1} of Σ.

More generally assume that $\mathcal{A} = \{F_n^i\}_{i=1}^r$ is an arrangement of subspaces of the form

$$F_n^i = D_i + SP^{n-d_i}(X) \quad \text{where} \quad D_i \in SP^{d_i}(X) \quad i = 1, \ldots, r. \tag{2}$$

Moreover, we assume that for each i the corresponding divisor D_i has the form

$$D_i = \alpha_1^i x_1 + \ldots + \alpha_k^i x_k$$

where all points $x_i \in X$ are distinct and fixed in advance while α_i are non-negative integers. The associated intersection poset has several useful interpretations. Let $J = \{j_1 < \ldots < j_m\}$ be a subsequence of $[k] = \{1, \ldots, k\}$. Since $K \in \bigcap_{\alpha=1}^m (D_{j_\alpha} + SP^{n-d_{j_\alpha}}(X))$ if and only if $K \in SP^n(X)$ and $D_{j_\alpha} \leq K$ for each $\alpha = 1, \ldots, m$ we

observe that $K \geq D_{j_1,j_2,\ldots,j_m} =: D_J$ where D_J is the least upper bound of divisors D_{j_α}, $\alpha = 1,\ldots,m$. Another description is in terms of *multisets* [30] or the associated monomials. Given an effective divisor $D = \alpha_1 x_1 + \ldots + \alpha_r x_r$, where points x_i are distinct and α_i are the corresponding multiplicities, the associated multiset (monomial) is $x_1^{\alpha_1} \ldots x_r^{\alpha_r}$. Then the intersection poset $P = P(\mathcal{A})$ can be described as the collection of all multisets in X of cardinality at most n which can be represented as unions of multisets associated to original divisors D_i. Yet another description arises if multisets $x_1^{\alpha_1} \ldots x_r^{\alpha^r}$ are interpreted as natural numbers $p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ where p_i are distinct prime integers. This shows that the combinatorics of intersections posets of arrangements of subspaces in $SP^n(X)$ is directly connected with the classical elementary number theory.

Caveat: All spaces we deal with are *admissible* in the sense of Definition 1.14. Elements of the intersection poset $P = P(\mathcal{A})$ are often denoted by p, q, r etc. but when we want to emphasize that they are actually divisors (multisets) we use the notation I, J, K, D_j etc.

Let $\mathcal{A}$ be an arrangement of subspaces described by (2). Let $\mathcal{D} : P \to Top$ be an associated diagram of spaces and inclusion maps, [35] [38]. Each $I \in P$ is of the form $I = \beta_1 x_1 + \ldots + \beta_k x_k$ where $|I| = \beta_1 + \ldots + \beta_k \leq n$. Hence,

$$\mathcal{D}(I) := \beta_1 x_1 + \ldots + \beta_k x_k + SP^{n-|I|}(X).$$

By Projection Lemma, [35] Lemma 4.5, or by Proposition 6.9. on page 49 in [3],

$$F_n = \bigcup \mathcal{A} \cong \textbf{colim } \mathcal{D} \simeq \textbf{hocolim } \mathcal{D}. \tag{3}$$

Definition 1.1. *For $A \subset SP^p(X)$ and $B \subset SP^q(X)$, the "Minkowski" sum $A + B$ is a subset of $SP^{p+q}(X)$ defined by $A + B := \{a + b \mid a \in A, b \in B\}$.*

Since X is an admissible space, Definition 1.14, there exists a closed, contractible set $C \supset \{x_1, \ldots, x_k\}$ such that the projection map $X \to X/C$ is a homotopy equivalence. Moreover, C contracts to a point $y \in C$ which can be prescribed in advance.

Define $\mathcal{E} : P \to Top$ to be the diagram of spaces and inclusion maps determined by

$$\mathcal{E}(I) := |I|y + SP^{n-|I|}(X) \quad \text{ for each } \quad I \in P.$$

Note that $\mathcal{E}(I)$ depends only on the order $|I|$ of the divisor I. We would like to show that

$$\textbf{hocolim } \mathcal{D} \simeq \textbf{hocolim } \mathcal{E}.$$

Since there does not exist an obvious map between these diagrams, we define a new diagram $\mathcal{C}$, a so called "ample space" diagram, which contains both $\mathcal{D}$ and $\mathcal{E}$ as subdiagrams.

Let $\mathcal{C} : P \to Top$ be the diagram of spaces and inclusion maps defined by

$$\mathcal{C}(I) := SP^{|I|}(C) + SP^{n-|I|}(X) \quad \text{ for each } \quad I \in P.$$

Proposition 1.2. *Let*

$$\alpha : \mathcal{D} \to \mathcal{C} \quad \textit{and} \quad \beta : \mathcal{E} \to \mathcal{C}$$

be the morphisms of diagrams where $\alpha_I : \mathcal{D}(I) \to \mathcal{C}(I)$ and $\beta_I : \mathcal{E}(I) \to \mathcal{C}(I)$ are obvious inclusions. Then α and β induce the homotopy equivalences of the corresponding homotopy colimits,

$$\textbf{hocolim } \mathcal{D} \xrightarrow{\hat{\alpha}} \textbf{hocolim } \mathcal{C} \xleftarrow{\hat{\beta}} \textbf{hocolim } \mathcal{E}.$$

Proof: The proposition is an immediate consequence of the Homotopy Lemma, [35] Lemma 4.6, and Proposition 1.13 from Section 1.5.

1.2 Steenrod's theorem for diagrams

In the previous section the calculation of the homology of the union of the arrangement $\mathcal{A} = \{D_i + SP^{n-d_i}(X)\}_{i=1}^r$ was reduced to the calculation of $H_*(\textbf{hocolim } \mathcal{E})$ for a diagram $\mathcal{E}$ of particularly simple form, cf. Definition 1.4.

Our objective in this section is to establish a decomposition result (Theorem 1.8) expressing the homology of **hocolim** $\mathcal{E}$ in simpler terms. Theorem 1.8 can be seen as an offspring and a generalization of the well known Steenrod's theorem,

$$H_*(SP^m(X); \mathbb{A}) \cong \bigoplus_{j=0}^{m} H_*(SP^j(X), SP^{j-1}; \mathbb{A})$$

where by definition $SP^{-1}(X) = \emptyset$ and $\mathbb{A}$ is an arbitrary Abelian group.

Definition 1.3. *Assume that $y \in X$ is a point fixed in advance. A* standard *inclusion*

$$e_{p,q} : SP^p(X) \hookrightarrow SP^q(X),$$

associated to a pair of integers $p \leq q$, is by definition the map defined by $e_{p,q}(Y) := (q-p)y + Y$ for each $Y \in SP^p(X)$.

Definition 1.4. *A diagram of spaces $\mathcal{D} : P \to Top$ is called* simple *if*

(a) *for each $p \in P$ there exists $\mu = \mu(p)$ such that $\mathcal{D}(p) = SP^{\mu}(X)$,*

(b) *the map $\mathcal{D}_{q,p} : SP^{\mu(p)}(X) \to SP^{\mu(q)}(X)$ is a standard inclusion for each pair $q \leq p$.*

The monotone function $\mu : P \to \mathbb{N}$ is called the rank *function of $\mathcal{D}$. If $\mu(p) = \mu(q)$ for each pair $p, q \in P$ we say that $\mathcal{D}$ is a* constant *diagram.*

Remark 1.5. *Note that a diagram of spaces $\mathcal{D} : P \to Top$ is simple if and only if there exists a strictly increasing sequence $m = c_0 < c_1 < \ldots < c_k = M$ and a strictly decreasing sequence $P_m = P_{c_0} \supset P_{c_1} \supset \ldots \supset P_{c_k} = P_M$ of ideals in P such that $\mathrm{Im}(\mu) = \{c_i\}_{i=0}^k$ and for each $p \in P_{c_i} \setminus P_{c_{i+1}}$, $\mathcal{D}(p) = SP^{c_i}(X)$.*

Definition 1.6. *Suppose that $\mathcal{E}_1$ and $\mathcal{E}_2$ are both simple diagrams over the same poset P and let μ_1 and μ_2 be the corresponding rank functions. We write $\mathcal{E}_1 \preccurlyeq \mathcal{E}_2$ and say that the diagram $\mathcal{E}_1$ is dominated by the diagram $\mathcal{E}_2$ if $\mu_1 \leq \mu_2$. We write $\mathcal{E}_1 \preccurlyeq_P \mathcal{E}_2$ and say that the diagram $\mathcal{E}_1$ is* properly *dominated by $\mathcal{E}_2$ if $\mu_1 = \min\{\mu_2, c\}$ for some constant $c \in \mathbb{N}$.*

It is obvious that both $\preccurlyeq$ and $\preccurlyeq_P$ are partial orders on the set of all simple diagrams over P. If either $\mathcal{E}_1 \preccurlyeq \mathcal{E}_2$ or $\mathcal{E}_1 \preccurlyeq_P \mathcal{E}_2$, there is a unique morphism $\alpha : \mathcal{E}_1 \to \mathcal{E}_2$ such that $\alpha_p : \mathcal{E}_1(p) \to \mathcal{E}_2(p)$ is a standard inclusion for each $p \in P$.

Proposition 1.7. *Suppose that $\mathcal{E}_1 \preccurlyeq_P \mathcal{E}_2 \preccurlyeq_P \mathcal{E}_3$ and let $\mathcal{E}_1 \xrightarrow{\alpha} \mathcal{E}_2 \xrightarrow{\beta} \mathcal{E}_3$ be the associated chain of morphisms. Suppose that*

$$X_{\mathcal{E}_1} \xrightarrow{\hat{\alpha}} X_{\mathcal{E}_2} \xrightarrow{\hat{\beta}} X_{\mathcal{E}_3}$$

*is the corresponding chain of homotopy colimits $X_{\mathcal{E}_i} :=$ **hocolim** $\mathcal{E}_i$, $i = 1, 2, 3$. Then the map*

$$H_*(X_{\mathcal{E}_2}, X_{\mathcal{E}_1}; \mathbb{A}) \longrightarrow H_*(X_{\mathcal{E}_3}, X_{\mathcal{E}_1}; \mathbb{A})$$

is injective and the associated long exact sequence of the triple splits.

Proof: It is sufficient to prove that the map

$$\hat{\alpha} : H_*(X_{\mathcal{D}}; \mathbb{A}) \longrightarrow H_*(X_{\mathcal{E}}; \mathbb{A}) \tag{4}$$

is injective for each pair of simple diagrams such that $\mathcal{D} \preccurlyeq_P \mathcal{E}$. Indeed, on applying this result on pairs $\mathcal{E}_1 \preccurlyeq_P \mathcal{E}_2$ and $\mathcal{E}_1 \preccurlyeq_P \mathcal{E}_3$ we obtain a commutative diagram with exact rows,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_*(X_{\mathcal{E}_1}) & \longrightarrow & H_*(X_{\mathcal{E}_2}) & \longrightarrow & H_*(X_{\mathcal{E}_2}, X_{\mathcal{E}_1}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_*(X_{\mathcal{E}_1}) & \longrightarrow & H_*(X_{\mathcal{E}_3}) & \longrightarrow & H_*(X_{\mathcal{E}_3}, X_{\mathcal{E}_1}) & \longrightarrow & 0
\end{array}$$

The first vertical arrow from the left is an isomorphism and the second is a monomorphism, so the result follows from a version of 5-lemma.

Let us observe that (4) is obvious if both $\mathcal{D}$ and $\mathcal{E}$ are constant simple diagrams in the sense that for some integers $m \leq n$, $\mathcal{D}(p) = SP^m(X)$ and $\mathcal{E}(p) = SP^n(X)$ for each $p \in P$. Indeed, in this case (4) reduces to the monomorphism

$$H_*(SP^m(X) \times \Delta(P)) \longrightarrow H_*(SP^n(X) \times \Delta(P))). \tag{5}$$

In light of the fact that by Steenrod's theorem $H_*(SP^m(X)) \to H_*(SP^n(X))$ is always a monomorphism to a direct summand of $H_*(SP^n(X))$, (5) follows from Künneth formula and the 5-lemma. Next we observe that (4) is true even if only $\mathcal{D}$ is a constant diagram. Indeed, let $\mathcal{F}$ be a constant simple diagram such that $\mathcal{E} \preccurlyeq \mathcal{F}$. Then the composition $\hat{\beta} \circ \hat{\alpha}$ in the diagram

$$H_*(X_{\mathcal{D}}) \xrightarrow{\hat{\alpha}} H_*(X_{\mathcal{E}}) \xrightarrow{\hat{\beta}} H_*(X_{\mathcal{F}})$$

is a monomorphism, hence $\hat{\alpha}$ alone is also a monomorphism.

The general case of (4) is established by induction on the size of the poset P. Let $\mathcal{C}$ be a maximal constant simple diagram over P such that $\mathcal{C} \preccurlyeq \mathcal{D}$. In other words if $\mathcal{D}(p) = SP^{\mu(p)}(X)$ for each $p \in P$ then $\mathcal{C}(p) := SP^m(X)$ where $m := \min\{\mu(p)\}_{p \in P}$. Let P' be a subposet of P defined by $P' := \{p \in P \mid \mu(p) > m\}$. Note that P' is actually an ideal in P. Define $\mathcal{C}', \mathcal{D}', \mathcal{E}'$ respectively as the restrictions of diagrams $\mathcal{C}, \mathcal{D}, \mathcal{E}$ on the subposet P'. Then by the excision axiom there is a commutative diagram of long exact sequences

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & H_*(X_{\mathcal{D}'}, X_{\mathcal{C}'}) & \longrightarrow & H_*(X_{\mathcal{E}'}, X_{\mathcal{C}'}) & \longrightarrow & H_*(X_{\mathcal{E}'}, X_{\mathcal{D}'}) & \longrightarrow & \ldots \\
 & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\
\ldots & \longrightarrow & H_*(X_{\mathcal{D}}, X_{\mathcal{C}}) & \longrightarrow & H_*(X_{\mathcal{E}}, X_{\mathcal{C}}) & \longrightarrow & H_*(X_{\mathcal{E}}, X_{\mathcal{D}}) & \longrightarrow & \ldots
\end{array} \tag{6}$$

The condition $\mathcal{C} \preccurlyeq_P \mathcal{D} \preccurlyeq_P \mathcal{E}$ implies $\mathcal{C}' \preccurlyeq_P \mathcal{D}' \preccurlyeq_P \mathcal{E}'$ and by the inductive assumption the first row splits. Hence there is a short exact sequence

$$0 \longrightarrow H_*(X_{\mathcal{D}}, X_{\mathcal{C}}) \xrightarrow{1\text{–}1} H_*(X_{\mathcal{E}}, X_{\mathcal{C}}) \longrightarrow H_*(X_{\mathcal{E}}, X_{\mathcal{D}}) \longrightarrow 0.$$

Finally, from the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_*(X_{\mathcal{C}}) & \xrightarrow{\cong} & H_*(X_{\mathcal{C}}) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_*(X_{\mathcal{D}}) & \xrightarrow{\hat{\alpha}} & H_*(X_{\mathcal{E}}) & \longrightarrow & H_*(X_{\mathcal{E}}, X_{\mathcal{D}}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_*(X_{\mathcal{D}}, X_{\mathcal{C}}) & \xrightarrow{1\text{–}1} & H_*(X_{\mathcal{E}}, X_{\mathcal{C}}) & \longrightarrow & H_*(X_{\mathcal{E}}, X_{\mathcal{D}}) & \longrightarrow & 0
\end{array}$$

we deduce that $\hat{\alpha}$ is a monomorphism. $\square$

Given a simple diagram $\mathcal{D} : P \to Top$, $p \mapsto SP^{\mu(p)}(X)$, let $m := \min\{\mu(p)\}_{p\in P}$ and $M := \max\{\mu(p)\}_{p\in P}$. Let us assemble the elements of the set $\{\mu(p)\}_{p\in P}$ into an increasing sequence

$$m = c_0 < c_1 < \ldots < c_k = M. \tag{7}$$

Define $P_j := \{p \in P \mid \mu(p) \geq j\}$. Note that

$$P = P_m \supseteq P_{m+1} \supseteq \ldots \supseteq P_M \tag{8}$$

is a decreasing sequence of ideals in P. In light of (7) we observe that

$$P_m = P_{c_0} \supset P_{c_1} \supset \ldots \supset P_{c_k} = P_M \tag{9}$$

is a subsequence of (8) obtained by removing the redundant posets.

Theorem 1.8. *Assume that $\mathcal{D} : P \to Top$ is a simple diagram where $\mu : P \to \mathbb{N}$ is the corresponding rank function, $m = \min\{\mu(p)\}_{p\in P}, M = \max\{\mu(p)\}_{p\in P}$ and $P_j := \{p \in P \mid \mu(p) \geq j\}$. Then the homology of $X_{\mathcal{D}} = \mathbf{hocolim}\ \mathcal{D}$ with coefficients in a group $\mathbb{A}$ admits the decomposition*

$$\begin{aligned} H_*(X_{\mathcal{D}}) &\cong H_*(SP^m(X) \times \Delta(P)) \bigoplus_{j=m+1}^{M} H_*((SP^j(X), SP^{j-1}(X)) \times \Delta(P_j)) \\ &\cong H_*(SP^m(X) \times \Delta(P)) \bigoplus_{p=1}^{k} H_*((SP^{c_p}(X), SP^{c_{p-1}}(X)) \times \Delta(P_{c_p})). \end{aligned} \tag{10}$$

Proof: The result is easily deduced from Proposition 1.7. Let

$$\mathcal{D}_0 \preccurlyeq_P \mathcal{D}_1 \preccurlyeq_P \ldots \preccurlyeq_P \mathcal{D}_{k-1} \preccurlyeq_P \mathcal{D}_k = \mathcal{D}$$

be the sequence of simple diagrams over P where $\mathcal{D}_j(p) = SP^{c_j}(X)$ if $p \in P_{c_j}$ and $\mathcal{D}_j(p) = \mathcal{D}(p)$ otherwise, while $(c_j)_{j=0}^k$ is the sequence defined in (7). In other words $\mathcal{D}_j$ is the simple diagram associated to the rank function $\mu_j = \min\{\mu, c_j\}$. By Proposition 1.7

$$H_*(X_{\mathcal{D}}) \cong H_*(X_{\mathcal{D}_0}) \oplus \bigoplus_{i=1}^{k} H_*(X_{\mathcal{D}_i}, X_{\mathcal{D}_{i-1}}). \tag{11}$$

Since $\mathcal{D}_0$ is a constant simple diagram we know that

$$H_*(X_{\mathcal{D}_0}) \cong H_*(SP^m(X) \times H_*(\Delta(P)). \tag{12}$$

Let $\mathcal{E}_i$ and $\mathcal{F}_i$ be constant simple diagrams over P_{c_i} such that $\mathcal{E}_i(p) = SP^{c_i}(X)$ and $\mathcal{F}_i(p) = SP^{c_{i-1}}(X)$ for each $p \in P_{c_i}$. By the excision axiom

$$H_*(X_{\mathcal{D}_i}, X_{\mathcal{D}_{i-1}}) \cong H_*(X_{\mathcal{E}_i}, X_{\mathcal{F}_i}) \cong H_*((SP^{c_i}(X), SP^{c_{i-1}}(X)) \times \Delta(P_{c_i})). \tag{13}$$

The formulas (10) follow from (11) (12) (13), the observation that

$$H_*((SP^{c_i}(X), SP^{c_{i-1}}(X)) \times \Delta(P_{c_i})) \cong \bigoplus_{j=c_{i-1}+1}^{c_i} H_*((SP^j(X), SP^{j-1}(X)) \times \Delta(P_j)) \tag{14}$$

and the fact that $P_j = P_{c_i}$ for each j in the interval $(c_{i-1}, c_i]$. $\square$

As an immediate consequence of Proposition 1.2, Theorem 1.8, the homotopy equivalence (3) and the fact that $\mathcal{E}$ is a simple diagram, we obtain the following result.

Theorem 1.9. *Suppose that $\mathcal{A} = \{F_n^i\}_{i=1}^r$ is a diagram of subspaces of $SP^n(X)$ where $F_n^i = D_i + SP^{n-|D_i|}(X)$. Let P be the associated intersection poset and $\mu : P \to \mathbb{N}$ the corresponding rank function. Define m and M respectively as the minimum and the maximum of the set $\mu(P) \subset \mathbb{N}$ and let $P_j := \mu^{-1}(\mathbb{N}_{\geq j})$. Then, for the homology with coefficients in a group $\mathbb{A}$,*

$$\begin{aligned} H_*(\bigcup \mathcal{A}) \quad &\cong \quad H_*(SP^m(X) \times \Delta(P)) \bigoplus \\ &\bigoplus_{j=m+1}^{M} H_*((SP^j(X), SP^{j-1}(X)) \times \Delta(P_j)). \end{aligned} \tag{15}$$

The following corollary of Theorem 1.9 is needed in the proof of Theorem 2.5.

Corollary 1.10. *Let $\mathcal{A} = \{F_n^i\}_{i=1}^k$ be an arrangement of subspaces in $SP^n(X)$ where $F_n^i := x_i + SP^{n-1}(X)$ and $x_1, \ldots, x_k$ are distinct points in X. Then, for the homology with rational coefficients,*

$$\begin{aligned} H_*(\bigcup \mathcal{A}; \mathbb{Q}) \quad &\cong \quad H_*(SP^{n-m}(X)) \otimes H_*(\Sigma^{m-1}) \bigoplus \\ &\bigoplus_{p=0}^{m-2} H_*(SP^{n-p-1}(X), SP^{n-p-2}(X)) \otimes H_*(\Sigma^p) \end{aligned} \tag{16}$$

where Σ^p is the p-skeleton of a simplex Σ with k vertices and $m := \min\{n, k\}$.

1.3 Category of simple diagrams

In this section we take a closer look at the category of simple diagrams and recast the main results of Section 1.1 in a form suitable for applications in Section 1.4. The emphasis is on functorial properties (naturality) of decompositions (10) and (15).

Let $\mathcal{P}os$ be the category of finite posets and monotone (increasing) maps. Let $\mathcal{R}ank$ be the category of abstract rank functions defined on finite posets. The objects of $Rank$ are monotone (decreasing) functions $\mu : P \to \mathbb{N}$. A monotone map $F : P \to Q$ defines a morphism $F : \mu \to \nu$ of two abstract rank functions μ and ν if $\mu \leq \nu \circ F$, i.e. if $\mu(p) \leq \nu \circ F(p)$ for each $p \in P$. If $F = 1_P$ is the identity map and $\mu \leq \nu$ then, as in Definition 1.6, we say that μ is dominated by ν and write $\mu \preccurlyeq \nu$.

The category *S-Diag* of simple diagrams is formally isomorphic to the category *Rank* of abstract rank functions. Objects of *S-Diag* are diagrams over finite posets which are simple in the sense of Definition 1.4. Suppose that $\mathcal{D} : P \to Top$ and $\mathcal{E} : Q \to Top$ are simple diagrams with the associated rank functions $\mu : P \to \mathbb{N}$ and $\nu : Q \to \mathbb{N}$. Then a morphism $(F, \alpha) : \mathcal{D} \to \mathcal{E}$ is defined if $F : P \to Q$ is a monotone map and $\mu \leq \nu \circ F$, in which case $\alpha(p) : \mathcal{D}(p) \to \mathcal{E}(F(p))$ is a standard inclusion in the sense of Definition 1.3. If F is clear from the context, for example in the case of an identity map, the corresponding morphism is simply denoted by α. A morphism (F, α) induces a continuous map $X_{(F,\alpha)} : X_{\mathcal{D}} \to X_{\mathcal{E}}$ of the corresponding homotopy colimits. Again we simplify and often write $\hat{\alpha}$ instead of $X_{(F,\alpha)}$ if $F : P \to Q$ is self-understood.

Suppose that $\mu : P \to \mathbb{N}$ is an object in *Rank* and let $\mathcal{D}$ be the associated simple diagram. Given $j \in \mathbb{N}$, let $\mu_j : P \to \mathbb{N}$ be defined by $\mu_j(p) = \min\{\mu(p), j\}$ for each $p \in P$. Define $\mathcal{D}_j$ as the simple diagram associated to μ_j and let $\alpha_j : \mathcal{D}_j \to \mathcal{D}$ be the associated morphism. The associated homotopy colimits $X_{\mathcal{D}_j}$ are subspaces of $X_{\mathcal{D}}$ which define a filtration

$$X_{\mathcal{D}_0} \subseteq X_{\mathcal{D}_1} \subseteq \ldots \subseteq X_{\mathcal{D}_j} \subseteq \ldots \subseteq X_{\mathcal{D}}. \tag{17}$$

Then in light of the decomposition (14), Theorem 1.8 can be rewritten as follows

Theorem 1.11. *Assume that $\mathcal{D} : P \to Top$ is a simple diagram where $\mu : P \to \mathbb{N}$ is the corresponding rank function. Let $P_j := \{p \in P \mid \mu(p) \geq j\}$. Then, for the homology with coefficients in an arbitrary group $\mathbb{A}$,*

$$H_*(X_\mathcal{D}; \mathbb{A}) \cong \bigoplus_{j=0}^{\infty} H_*((SP^j(X), SP^{j-1}(X)) \times \Delta(P_j)) \tag{18}$$

where by definition $SP^{-1}(X) := \emptyset =: \Delta(\emptyset)$.

The following proposition essentially claims that the decomposition (18) is natural with respect to morphisms $(F, \alpha) : \mathcal{D} \to \mathcal{E}$ in the category *S-Diag*.

Proposition 1.12. *Suppose that $(F, \alpha) : \mathcal{D} \to \mathcal{E}$ is a morphism of two simple diagrams and let $\hat{\alpha} : X_\mathcal{D} \to X_\mathcal{E}$ be the induced map of the associated homotopy colimits. If $\{X_{\mathcal{D}_j}\}_{j=0}^{\infty}$ and $\{X_{\mathcal{E}_j}\}_{j=0}^{\infty}$ are the filtrations of $X_\mathcal{D}$ and $X_\mathcal{E}$ described by (17) then $\hat{\alpha}(X_{\mathcal{D}_j}) \subseteq X_{\mathcal{E}_j}$. It follows that there exists a homomorphism*

$$H_*((SP^j(X), SP^{j-1}(X)) \times \Delta(P_j)) \longrightarrow H_*((SP^j(X), SP^{j-1}(X)) \times \Delta(Q_j)) \tag{19}$$

of the corresponding terms in the decompositions (18) of $H_*(X_\mathcal{D})$ and $H_*(X_\mathcal{E})$ respectively. The homomorphism (19) is induced by the map $\Delta(F_j) : \Delta(P_j) \to \Delta(Q_j)$ where $F_j : P_j \to Q_j$ is the restriction of F to P_j.

Proof: The condition $\mu \leq \nu \circ F$ implies

$$\mu_j = \min\{\mu, j\} \leq \min\{\nu \circ F, j\} = \min\{\nu, j\} \circ F = \nu_j \circ F.$$

It follows that there is a morphism in *S-Diag* of diagrams $\mathcal{D}_j$ and $\mathcal{E}_j$ and an associated continuous map $\hat{\alpha}_j : X_{\mathcal{D}_j} \to X_{\mathcal{E}_j}$ of the corresponding homotopy colimits. Moreover, there is a ladder of commutative diagrams

$$\begin{array}{ccccccccc} \ldots \longrightarrow & X_{\mathcal{D}_{j-1}} & \longrightarrow & X_{\mathcal{D}_j} & \longrightarrow & X_{\mathcal{D}_{j+1}} & \longrightarrow \ldots \longrightarrow & X_\mathcal{D} \\ & {\scriptstyle \hat{\alpha}_{j-1}} \downarrow & & {\scriptstyle \hat{\alpha}_j} \downarrow & & {\scriptstyle \hat{\alpha}_{j+1}} \downarrow & & \downarrow {\scriptstyle \hat{\alpha}} \\ \ldots \longrightarrow & X_{\mathcal{E}_{j-1}} & \longrightarrow & X_{\mathcal{E}_j} & \longrightarrow & X_{\mathcal{E}_{j+1}} & \longrightarrow \ldots \longrightarrow & X_\mathcal{E} \end{array} \tag{20}$$

where the horizontal maps are the inclusions coming from the filtration (17). It follows that there is a homomorphism

$$H_*(X_\mathcal{D}; \mathbb{A}) \cong \bigoplus_{j=0}^{\infty} H_*(X_{\mathcal{D}_j}, X_{\mathcal{D}_{j-1}}; \mathbb{A}) \longrightarrow \bigoplus_{j=0}^{\infty} H_*(X_{\mathcal{E}_j}, X_{\mathcal{E}_{j-1}}; \mathbb{A}) \cong H_*(X_\mathcal{E}; \mathbb{A}) \tag{21}$$

where $X_{\mathcal{D}_{-1}} = X_{\mathcal{D}_{-1}} = \emptyset$. The final part of Proposition 1.12 follows from the naturality of the excision operation or more precisely from the naturality of the diagram (6). □

1.4 Homology of the complement $SP^n(M_g) \setminus \bigcup \mathcal{A}$

In this section we focus our attention on the homology of the complement $SP^n(M_g) \setminus F_n$ of the arrangement $\mathcal{A}$ where M_g is an orientable surface of genus g. We will be primarily interested in the homology with rational coefficients. By Poincaré duality, the evaluation of these groups is equivalent to the evaluation of the homology of the pair $H_*((SP^n(M_g), F_n); \mathbb{Q})$ which is directly related to the evaluation of the kernel A_j and the cokernel B_j of the homomorphism

$$H_j(\bigcup \mathcal{A}) \longrightarrow H_j(SP^n(X)) \tag{22}$$

for an arrangement $\mathcal{A}$ in $SP^n(X)$ where X is an *admissible* space.

Given a map of diagrams $\alpha : \mathcal{E}_1 \to \mathcal{E}_2$, there is a commutative square

$$\begin{array}{ccc} \mathbf{hocolim}\ \mathcal{E}_1 & \longrightarrow & \mathbf{hocolim}\ \mathcal{E}_2 \\ \downarrow & & \downarrow \\ \mathbf{colim}\ \mathcal{E}_1 & \longrightarrow & \mathbf{colim}\ \mathcal{E}_2 \end{array} \tag{23}$$

In light of the fact that for each arrangement of subspaces $\mathcal{F}$ and the corresponding diagram of spaces $\mathcal{D}_{\mathcal{F}}$

$$\bigcup \mathcal{F} \cong \mathbf{colim}\ \mathcal{D}_{\mathcal{F}},$$

the square (23) allows us to compare spaces $F_n = \bigcup \mathcal{A}$ and $SP^n(X)$ by comparing the associated diagrams.

The diagrams $\mathcal{C}, \mathcal{D}, \mathcal{E}$, the associated intersection poset P etc. have the same meaning as in Section 1.1. Let $\hat{P} := P \cup \{0\}$ be the poset P with added a possibly new minimum element 0. Let $\mathcal{D}_0 : \hat{P} \to Top$ be a diagram of spaces and inclusion maps defined by $\mathcal{D}_0(0) = SP^n(X)$ and $\mathcal{D}_0(p) = \mathcal{D}(p)$ for each $p \in P$. Similarly, $\mathcal{C}_0$ and $\mathcal{E}_0$ are defined as the diagrams over $\hat{P}$ which extend the diagrams $\mathcal{C}$ and $\mathcal{E}$ respectively, such that $\mathcal{C}_0(0) = \mathcal{E}_3(0) = SP^n(X)$. The inclusion $e : P \to \hat{P}$ extends to the corresponding morphisms of diagrams

$$\mathcal{C} \xrightarrow{\gamma} \mathcal{C}_3 \qquad \mathcal{D} \xrightarrow{\delta} \mathcal{D}_0 \qquad \mathcal{E} \xrightarrow{\eta} \mathcal{E}_1$$

Moreover, there is a commutative diagram

$$\begin{array}{ccccc} X_{\mathcal{D}} & \xrightarrow{\hat{\alpha}} & X_{\mathcal{C}} & \xleftarrow{\hat{\beta}} & X_{\mathcal{E}} \\ {\scriptstyle\hat{\delta}}\downarrow & & {\scriptstyle\hat{\gamma}}\downarrow & & {\scriptstyle\hat{\eta}}\downarrow \\ X_{\mathcal{D}_0} & \xrightarrow{\hat{\alpha}_0} & X_{\mathcal{C}_0} & \xleftarrow{\hat{\beta}_0} & X_{\mathcal{E}_0} \end{array} \tag{24}$$

where all horizontal maps are homotopy equivalences and as before $X_{\mathcal{G}} = \mathbf{hocolim}\ \mathcal{G}$. An instance of the commutative diagram (23) is the following square

$$\begin{array}{ccc} X_{\mathcal{D}} & \xrightarrow{\hat{\delta}} & X_{\mathcal{D}_0} \\ \downarrow & & \downarrow \\ \bigcup \mathcal{A} & \longrightarrow & SP^n(M_g) \end{array} \tag{25}$$

By Projection Lemma, [35] Lemma 4.5, the vertical arrows in this diagram are homotopy equivalences. In light of (24) we conclude that the groups A_d and B_d are respectively the kernel and the cokernel of the map

$$H_d(X_{\mathcal{E}}) \xrightarrow{\eta_*} H_d(X_{\mathcal{E}_0}). \tag{26}$$

By Theorem 1.11 each of the groups $H_d(X_{\mathcal{E}})$ and $H_d(X_{\mathcal{E}_0})$ admits a direct sum decomposition of the form (18). By Proposition 1.12 this decomposition is natural and the corresponding terms are mapped to each other. More precisely there is a homomorphism

$$H_*((SP^j(X), SP^{j-1}(X)) \times \Delta(P_j)) \longrightarrow H_*((SP^j(X), SP^{j-1}(X)) \times \Delta(\hat{P}_j)) \tag{27}$$

induced by the map $\Delta(e_j) : \Delta(P_j) \to \Delta(\hat{P}_j)$ where e_j is an inclusion map.

Note that $\Delta(\hat{P}_j)$ is contractible whenever $\hat{P}_j \neq \emptyset$. It immediately follows that for the homology with *rational coefficients*, $A_d \cong A_d^{(1)} \oplus A_d^{(2)}$ where

$$A_d^{(1)} \cong \bigoplus_{j=0}^{+\infty} \bigoplus_{p+q=d}^{q>0} H_p(SP^j(X), SP^{j-1}(X)) \otimes H_q(\Delta(P_j)) \tag{28}$$

$$A_d^{(2)} \cong \bigoplus_{j=9}^{\infty} H_d(SP^j(X), SP^{j-1}) \otimes \Omega_j \tag{29}$$

where $\Omega_j := \mathrm{Ker}\{H_0(\Delta(P_j)) \to H_0(\Delta(\hat{P}_j))\} = \tilde{H}_0(\Delta(P_j))$ is the reduced, 0-dimensional homology of P_j. Similarly,

$$B_d \cong H_d(SP^n(X), SP^M(X)) \tag{30}$$

where $M = \max\{\mu(p)\}_{p\in P}$ is the maximum of the associated rank function.

1.5 An auxiliary result

Proposition 1.13. *Suppose that X is an* admissible *space in the sense of Definition 1.14. Given a finite set $F = \{y, y_1, \dots, y_d\}$ in Z, let C be an associated contractible superset of F. Let $Y = y_1 + y_2 + \dots + y_d \in SP^d(X)$. Then the inclusion map*

$$Y + SP^{n-d}(X) \xrightarrow{\alpha} SP^d(C) + SP^{n-d}(X) \tag{31}$$

is a homotopy equivalence.

Proof: We prove the lemma first in the special case when all points y_i coincide. More precisely we prove that the map $d\,y + SP^{n-d}(X) \hookrightarrow SP^d(C) + SP^{n-d}(X)$ is a homotopy equivalence.

It follows from the assumptions on X, C and $y \in C$ that there exists a homotopy $h : X \times I \to X$, keeping y fixed and C invariant, such that $p : X \to X$, defined by $p(x) := h(x,1)$ satisfies the condition $p(C) = \{y\}$ and $1_X = h(\cdot, 0)$ is the identity map.

Let $H : SP^n(X) \times I \to SP^n(X)$ be the homotopy induced on $SP^n(X)$ by h. In other words if $Z = z_1 + \dots + z_n \in SP^n(X)$, then $H(Z,t) := h(z_1,t) + \dots + h(z_n,t)$. We observe that both $V := d\,y + SP^{n-d}(X)$ and $W := SP^d(C) + SP^{n-d}(X)$ are H-invariant, so the restrictions of H on these subspaces define the homotopies $H^1 : V \times I \to V$ and $H^2 : W \times I \to W$. The map $p : X \to X$ induces a map from $SP^n(X)$ to $SP^n(X)$ which restrict to a map $\hat{p} : W \to V$. It turns out that $\hat{p}$ is a homotopy inverse to the inclusion $i : V \hookrightarrow W$. Indeed,

$$H^1 : 1_V \simeq \hat{p} \circ i \qquad \text{and} \qquad H^2 : 1_W \simeq i \circ \hat{p}\,.$$

Now we turn to the case of a general divisor $Y = y_1 + \dots + y_d \in SP^d(X)$ and the associated inclusion map $Y + SP^{n-d}(X) \xrightarrow{\alpha} SP^d(C) + SP^{n-d}(X)$. Let $\phi : d\,y + SP^{n-d}(X) \to Y + SP^{n-d}(X)$ be the homeomorphism defined by $\phi(d\,a + Z) = Y + Z$.

The key observation is the following equality

$$\hat{p} \circ \alpha \circ \phi = \hat{p} \circ i.$$

Since $\hat{p}$ and i are homotopy equivalences and ϕ is a homeomorphism, we conclude that α is also a homotopy equivalence. □

Definition 1.14. *A space X is called* admissible *if for each finite collection of points $F = \{y, x_1, \dots, x_k\} \subset X$, there exists a subspace $C \subset X$ such that*

(a) C contains F as a subset,

(b) C can be continuously deformed to y inside C keeping the point y fixed, i.e. $1_C \simeq c_y\,(\mathrm{rel}\ y)$ where $c_y(x) = y$ for each $x \in C$,

(c) the inclusion map $i : C \hookrightarrow X$ is a closed cofibration.

Remark 1.15. All connected spaces that can be triangulated are *admissible* in the sense of Definition 1.14. This is a very large class including all connected CW-complexes or connected semi-algebraic sets.

2 Applications

2.1 End spaces

Definition 2.1. *Suppose that Y is a locally compact, Hausdorff space. Let $\mathcal{P} = (\mathcal{K}, \subseteq)$ be the poset of all compact subspaces of Y. Let $\mathcal{D} : \mathcal{P} \to Top$ be the diagram of topological spaces over $\mathcal{P}$ defined by $\mathcal{D}(K) := Y \setminus K$ for $Q \in \mathcal{M}$. The* end space $e(Y)$ *of Y is by definition the homotopy limit of the diagram $\mathcal{D}$,*

$$e(Y) := \textbf{ holim } \mathcal{D}.$$

The reference [21] is recommended as a valuable source of information about the general theory, the history, and some of the latest applications of end spaces. The end space is obviously a topological invariant of Y, that is if Y and Y' are two locally compact, Hausdorff spaces such that $Y \cong Y'$ then the associated end spaces $e(Y)$ and $e(Y')$ are also homeomorphic. Consequently the homotopy type of $e(Y)$, its homology etc. are homeomorphism invariants of Y. If Y admits a cofinal sequence

$$K_0 \subseteq K_1 \subseteq \ldots \subseteq K_m \subseteq \ldots$$

of compact sets in Y, i.e. a sequence such that $Y = \bigcup_{m=0}^{\infty} K_m$, then

$$e(Y) \simeq \textbf{holim}_{m \mapsto \infty} Y \setminus K_m.$$

The following proposition allows us to "compute" the end space in the case the inclusion map $K_m \hookrightarrow K_{m+1}$ is a homotopy equivalence for each m.

Proposition 2.2. *Let $\mathcal{E} : \mathbb{N} \to Top$ be a diagram of topological spaces over $\mathbb{N}$ such that $\mathcal{D}(m) \to \mathcal{D}(m+1)$ is a homotopy equivalence for each m. Then*

$$\textbf{holim } \mathcal{E} \simeq \mathcal{D}(0).$$

As an illustration here is a computation of the (stable) homotopy type of the end space of some interesting spaces.

Proposition 2.3. *Let $Y = S^n \setminus X$ where X is a closed set in the sphere S^n. Let $D_n(A)$ be the stable homotopy type of the geometric dual of $A \subset S^n$,* [29]. *Then the end space of Y is stably equivalent to the geometric dual of the disjoint sum $X \cup D_n(X)$,*

$$e(Y) \simeq_P D_n(X \cup D_n(X)) \simeq_P S^{n-1} \vee X \vee D_n(X).$$

Proof: In light of the formula

$$D_n(A \cup B) \simeq_P D_n(S^0 \vee A \vee B) \simeq_P S^{n-1} \vee D_n(A) \vee D_n(B)$$

where A and B are disjoint, compact subsets of S^n, it is sufficient to observe that compact sets in $Y = S^n \setminus X$ which are deformation retracts of Y are cofinal in the poset of all compact subsets of Y.

2.2 End (co)homology groups

The diagram $\mathcal{D} : \mathcal{P} \to Top$ from Definition 2.1, in combination with standard functors, provides other interesting invariants or the homeomorphism type of Y.

Definition 2.4. *Let $\phi : Top \to Ab$ be a covariant (contravariant) functor from spaces to Abelian groups. Let $\phi(\mathcal{D}) : \mathcal{P} \to Ab$ be the associated (co)diagram defined by the correspondence $K \mapsto \phi(Y \setminus K)$. The (co)limit of $\phi(\mathcal{D})$ is an Abelian group which is called the end ϕ-group associated to Y. If ϕ is a homology (cohomology) functor with rational coefficients, then $\phi(\mathcal{D})$ is denoted by $\mathcal{H}_*$ and $\mathcal{H}^*$ respectively. In other words $\mathcal{H}_*(K) := H_*(Y \setminus K; \mathbb{Q})$ and $\mathcal{H}^*(K) := H^*(Y \setminus K; \mathbb{Q})$. The associated* end (co)homology groups *are defined by*

$$E_*(Y) := \mathbf{lim}\ \mathcal{H}_*, \qquad E^*(Y) := \mathbf{colim}\ \mathcal{H}^*.$$

In the following theorem we compute the group $E^*(SP^n(M_{g,k}))$ where $M_{g,k}$ is the orientable surface of genus g punctured at k distinct points. Sometimes we call the surface $M_{g,k}$ a (g,k)-amoeba, see Figure 1 where the amoebas $M_{1,3}$ and $M_{2,1}$ are shown. Note that although the amoebas $M_{g,k}$ and $M_{g',k'}$ are homeomorphic if and only if $(g,k) = (g',k')$, they have the same homotopy type if and only if $2g + k = 2g' + k'$.

Theorem 2.5. *Let $M_{g,k} = M_g \setminus \{x_1, \ldots, x_k\}$ be a (g,k)-amoeba, i.e. the Riemann surface of genus g with k distinct points removed. Let $SP^n(M_{g,k})$ be the associated symmetric product. If*

$$E^p(SP^n(M_{g,k})) := \mathbf{colim}\ \mathcal{H}^p = \mathbf{colim}_{W \in \mathcal{P}} H^p(QP^n(M_{g,k}) \setminus K; \mathbb{Q})$$

is the associated p-dimensional end cohomology group then

$$\mathrm{rank}(E^p(SP^n(M_{g,k}))) = \begin{cases} \binom{2g+k-1}{p}, & p \leq n-2 \\ \binom{2g+k}{n} - \binom{2g}{n}, & p = n-1 \text{ or } p = n \\ \binom{2g+k-1}{2n-1-p}, & p \geq n+2\,. \end{cases} \tag{32}$$

Proof: Let us choose a local metric in the vicinity of each point x_i, say by choosing a local coordinate system in the neighborhood of each of these points. Let $V_m^i = V(x_i, \frac{1}{m})$ be an open disc in M_g with the center at x_i of radius $\frac{1}{m}$, defined relative to the chosen metric. Let $W_m^i := V_m^i \setminus \{x_i\}$ be the corresponding punctured disc. Then $C_m = M_g \setminus \bigcup_{i=1}^k\ V(x_i, \frac{1}{m})$ is a compact subset in $M_{g,k}$ and $K_m := SP^n(C_m)$ is a compact subset in $SP^n(M_{g,k})$. The sequence $\{K_m\}_{m=1}^\infty$ is cofinal in the poset $\mathcal{P}$ of all compact subsets in $Y = SP^n(M_{g,k})$ since $\bigcup_{m=1}^{+\infty}\ K_m = SP^n(M_{g,k})$. Note that $Y \setminus K_m$ is described as the space of all divisors $D \in SP^n(M_{g,k})$ such that $D \cap V_m^i \neq \emptyset$ for some i.

Claim 1: The inclusion $Y \setminus K_{m+1} \hookrightarrow Y \setminus K_m$ induces an isomorphism $H^*(Y \setminus K_m; \mathbb{Q}) \to H^*(Y \setminus K_{m+1}; \mathbb{Q})$ of the associated cohomology groups.

The claim is an easy consequence of Poincaré duality. Indeed, both Y and its compact subsets K_m and K_{m+1} are identified with the corresponding subsets of the manifold $SP^n(M_g)$. Let $F_n := SP^n(M_g) \setminus Y$ be the subspace of all divisors $D \in SP^n(M_g)$ such that $D \cap \{x_1, \ldots, x_k\} \neq \emptyset$. Equivalently,

$$F_n := \bigcup_{j=1}^k (x_j + SP^{n-1}(M_g)).$$

There is a duality isomorphism

$$H^*(Y \setminus K_j) \to H_{2n-*}(SP^n(M_g); F_n \cup K_j) \tag{33}$$

natural with respect the inclusions $K_j \hookrightarrow K_{j+1}$. Then the claim follows from the five lemma and the fact that K_m is a deformation retract of K_{m+1}, which in turn implies the isomorphism $H_*(F_n \cup K_m) \to H_*(F_n \cup K_{m+1})$.

A consequence of the claim is the isomorphism

$$E^p(SP^n(M_{g,k})) \cong H^p(Y \setminus K_m) \cong H_{2n-p}(SP^n(M_g); F_n \cup K_m). \tag{34}$$

Hence, in order to complete the proof of the theorem it is sufficient to compute the group $H_d(SP^n(M_g); F_n \cup K_m)$.

Proposition 2.6. *Let $\Delta_d^{n,g}$ be the rank of the group $H_d(SP^n(M_g); F_n \cup K_m) \cong H^{2n-d}(SP^n(M_g) \setminus F_n \cup K_m)$. Then,*

$$\Delta_d^{n,g} = \begin{cases} \binom{2g+k-1}{d-1}, & d \leq n-1 \\ \binom{2g+k}{n} - \binom{2g}{n}, & d = n \text{ or } d = n+1 \\ \binom{2g+k-1}{2n-d}, & d \geq n+2\,. \end{cases} \tag{35}$$

Proof: The spaces F_n and K_m are disjoint compact subspaces of $SP^n(M_g)$ and $K_m \simeq SP^n(M_{g,k})$ has the homotopy type of an n-dimensional CW-complex. Consequently, the exact sequence of the pair $(SP^n(M_g), F_n \cup K_m)$ has the form

$$\begin{array}{ccccccc} H_d(F_n) \oplus H_d(K_m) & \xrightarrow{\Lambda_d} & H_d(SP^n(M_g)) & \longrightarrow & H_d(SP^n(M_g); F_n \cup K_m) & \longrightarrow \\ H_{d-1}(F_n) \oplus H_{d-1}(K_m) & \xrightarrow{\Lambda_{d-1}} & H_{d-1}(SP^n(M_g)) & \longrightarrow & \dots & \end{array} \tag{36}$$

An immediate consequence of (36) is the isomorphism

$$H_d(SP^n(M_g); F_n \cup K_m) \cong A_{d-1} \oplus B_d$$

where $A_d := \mathrm{Ker}(\Lambda_d)$ and $B_d := \mathrm{Coker}(\Lambda_d)$. Note that $\Lambda_d = \alpha_d + \beta_d$ where $\alpha_d : H_d(F_n) \to H_d(SP^n(M_g))$ and $\beta_d : H_d(SP^n(M_{g,k})) \to H_d(SP^n(M_g))$ are the homomorphisms induced by the inclusions $F_n \hookrightarrow SP^n(M_g)$ and $SP^n(M_{g,k}) \hookrightarrow SP^n(M_g)$.

Claim 2: Suppose that $\alpha : A \to C$ and $\beta : B \to C$ a linear maps of vector spaces. If $\alpha + \beta : A \oplus B \to C$ is the map defined by $(\alpha+\beta)(x \oplus y) := \alpha(x) + \beta(y)$ then

$$|\mathrm{Ker}(\alpha+\beta)| = |\mathrm{Ker}(\alpha)| + |\mathrm{Ker}(\beta)| + |\mathrm{Im}(\alpha) \cap \mathrm{Im}(\beta)| \tag{37}$$

where $|V|$ is the dimension of V.

An immediate consequence of the Claim 2 is the following equation

$$\begin{array}{rcl} \Delta_d^{n,g} & = & |\mathrm{Ker}(\alpha_{d-1})| + |\mathrm{Ker}(\beta_{d-1})| + |\mathrm{Im}(\alpha_{d-1} \cap \mathrm{Im}(\beta_{d-1})| + \\ & + & \Theta_d^{n,g} - |\mathrm{Im}(\alpha_d)| - |\mathrm{Im}(\beta_d)| + |\mathrm{Im}(\alpha_d) \cap \mathrm{Im}(\beta_d)| \end{array} \tag{38}$$

where $\Delta_d^{n,g} := |A_{d-1}| + |B_d|$ and $\Theta_d^{n,g} := |H_d(SP^n(M_g))|$. The long exact sequence of the pair $(SP^n(M_g), F_n)$ and the Poincaré duality imply that

$$\begin{array}{rcl} |\mathrm{Ker}(\alpha_{d-1})| + \Theta_d^{n,g} - |\mathrm{Im}(\alpha_d)| & = & |H_d(SP^n(M_g); F_n)| \;\; = \\ |H^{2n-d}(SP^n(M_g) \setminus F_n)| & = & \Phi_d^{n,g,k} \end{array} \tag{39}$$

where $\Phi_d^{n,g,k} = \binom{2g+k-1}{2n-d}$ if $n \leq d$ and 0 otherwise. It follows from equation (39), Proposition 3.1 and Proposition 3.2 that in order to compute the quantity $\Delta_d^{n,g}$ we need to discuss the four cases $d \leq n-1, d = n, d = n+1$ and $d \geq n+2$. The rest is an elementary calculation. This completes the proof of both Proposition 2.6 and Theorem 2.5. □

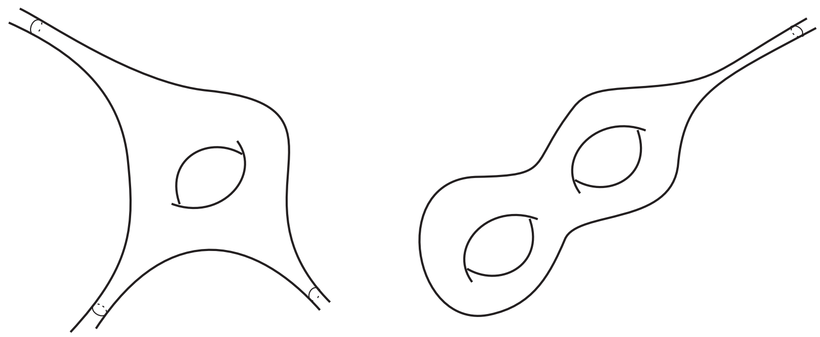

Figure 1: $M_{1,3}$ and $M_{2,1}$

2.3 Commutative $(m+k,m)$-groups

A commutative $(m+k,m)$-groupoid is a pair (X,μ) where the "multiplication" μ is a map $\mu : SP^{m+k}(X) \to SP^m(X)$. The operation μ is associative if for each $c \in SP^{m+2k}(X)$ and each representation $c = a+b$, where $a \in SP^{m+k}(X)$ and $b \in SP^k(X)$, the result $\mu(\mu(a) * b)$ is always the same, i.e. independent from the particular choice of a and b in the representation $c = a+b$. A commutative and associative $(m+k,m)$-groupoid is a $(m+k,m)$-group if the equation $\mu(x+a) = b$ has a solution $x \in SP^m(X)$ for each $a \in SP^k(X)$ and $b \in SP^m(X)$. The $(2,1)$-groups are essentially the groups in the usual sense of the word. If X is a topological space then (X,μ) is a topological $(m+k,m)$-group if it is a $(m+k,m)$-group and the map $\mu : SP^{m+k}(X) \to SP^m(X)$ is continuous.

For the motivation and other information about commutative $(m+k,m)$-groups the reader is referred to [33], [32]. The only known surfaces that support the structure of a $(m+k,m)$-group for $(m+k,m) \neq (2,1)$ are of the form $\mathbb{C}\backslash A$ where A is a finite set. It was proved in [32], see also Theorem 6.1 in [33] that if (M,μ) is a locally Euclidean, topological, commutative, $(m+k,m)$-group then M must be an orientable 2-manifold. Moreover, a 2-manifold that admits the structure of a commutative $(m+k,m)$-group satisfies a strong necessary condition that the symmetric power $SP^m(M) := M^m/S_m$ is of the form $\mathbb{R}^u \times (S^1)^v$.

This was a motivation for the authors to formulate in [9] the following problems

(A) To what extent is the topology of a surface M determined by the topology of its symmetric product $SP^m(M)$ for a given m?

(B) Are there examples of non-homeomorphic (open) surfaces M and N such that the associated symmetric products $SP^m(M)$ and $SP^m(N)$ are homeomorphic?

These problems are particularly interesting for the so called (g,k)-amoebas $M_{g,k}$, the surfaces defined by $M_{g,k} := M_g \setminus \{x_1,\dots,x_k\}$ where M_g is the Riemann surface of genus g, see Figure 1.

Since the end homology groups are invariants of homeomorphism types, a consequence of Theorem 2.5 is that both the genus g and the number k of points removed can be recovered from the knowledge of the end space $e(SP^n(M_{g,k}))$ in the case $2g \geq n$ (question (A)). As a corollary we obtain the main result of [9] (Theorem 1.1) which says that there exist open, orientable surfaces M and N such that the associated symmetric products $SP^m(M)$ and $SP^m(N)$ are not homeomorphic although they have the same homotopy type. More precisely, this is always true if $M = M_{g,k}$ and $N = M_{g',k'}$ $(k,k' \geq 1)$ and

- $2g+k = 2g'+k'$,

- $g \neq g'$ and $\max\{g, g'\} \geq m/2$.

This result puts some restrictions on potential examples asked for in question (B). Note that this result was proved in [9] by completely different methods based on the evaluation of the signature of general symmetric products $SP_G(M_{g,k}) := (M_{g,k})^n/G$ where G is a subgroup of S_n. The fact that two different methods led to the exactly same necessary condition $2g \geq n$ is intriguing and may be an indication that the answer to the question (B) is in fact positive. This however remains an interesting open problem which can be restated as follows.

Question: Could it be that some symmetric powers of amoebas $M_{1,3}$ and $M_{2,1}$ or more generally amoebas $M_{g,k}$ and $M_{g',k'}$ where $(g,k) \neq (g',k')$ but $2g+k = 2g'+k'$ are actually homeomorphic?

3 Appendix

3.1 Homology of symmetric products

It is well known [26] [22] that the Pontriagin algebra of $SP^\infty(M_g)$ has the form

$$H_*(SP^\infty(M_g); \mathbb{A}) \cong \Lambda(e_1, e_2, \dots, e_{2g}) \otimes \Gamma[M] \tag{40}$$

where $\Lambda(e_1, \dots, e_{2g})$ is an exterior algebra and $\Gamma[M]$ is the divided power algebra with generators γ_k, $k = 1, 2, \dots$. On tensoring with $\mathbb{Q}$ one has an isomorphism

$$H_*(SP^\infty(M_g); \mathbb{Q}) \cong \Lambda(e_1, e_2, \dots, e_{2g}) \otimes \mathbb{Q}[\gamma] \tag{41}$$

where $\mathbb{Q}[\gamma]$ is a polynomial algebra. The following classes

$$E_{(I,q)} = e_I\gamma^q = e_{i_1}e_{i_2}\dots e_{i_p} \cdot \gamma^q,\ I = \{i_1 < i_2 < \dots < i_p\}$$

form an additive basis of (41). Moreover,

$$H_*(SP^\infty(M_g)) \cong \bigoplus_{m\in\mathbb{N}} H_*(SP^m(M_g), SP^{m-1}(M_g)) \tag{42}$$

and $H_*(SP^m(M_g))$ is the subgroup spanned by all classes $E_{(I,q)} = e_{i_1}e_{i_2}\dots e_{i_p}\gamma^q$ where $p + q \leq m$.

Recall [26] [22] that by a classical result of Steenrod, the decomposition (42) holds for all connected CW-complexes X, in particular $H_*(SP^m(X); \mathbb{A}) \to H_*(SP^n(X); \mathbb{A})$ is always a monomorphism for $m \leq n$. Similarly, the isomorphism (41) can be seen as an instance of the celebrated result of Dold and Thom [15] which says that infinite symmetric products admit a decomposition into a product of Eilenberg-Mac Lane spaces

$$SP^\infty(X) \simeq \prod_{\nu\geq 0} K(\tilde{H}_\nu; \mathbb{A}), \nu). \tag{43}$$

Suppose that $Y = \bigvee_{j=1}^m S^1_j$ is a wedge of m circles. Then it is not difficult to show directly that

$$H_*(SP^\infty(Y; \mathbb{Q})) \cong \Lambda(e_1, \dots, e_m). \tag{44}$$

Since $M_{g,k} = M_g \setminus \{x_1, \dots, x_k\} \simeq \bigvee_{j=1}^{2g+k-1} S^1_j$ we conclude that

$$H_*(SP^\infty(M_{g,k}); \mathbb{A}) \cong \Lambda(e_1, e_2, \dots, e_{2g+k-1}). \tag{45}$$

The group $H_*(SP^m(M_{g,k}); \mathbb{Q})$ is generated by the classes $F_I = e_{i_1}\dots e_{i_p}$ where $I = \{i_1, \dots, i_p\} \subset [2g+k-1]$ and $p \leq m$. Moreover, the generators can be chosen so

that $\beta_*(F_I) = E_{I,0} = e_I$ if $I \subset [2g]$ and 0 otherwise where $\beta_* : H_*(SP^m(M_{g,k});\mathbb{Q}) \to H_*(SP^n(M_g);\mathbb{Q})$ is the map induced by the inclusion $SP^m(M_{g,k}) \hookrightarrow SP^m(M_g)$. As an immediate consequence we have the following result needed in the proof of Theorem 2.5 in Section 2. As before, $|V|$ is the rank of a vector space V.

Proposition 3.1.

$$\begin{aligned} |\mathrm{Ker}(\beta_d)| &= \begin{cases} \binom{2g+k-1}{d} - \binom{2g}{d}, & d \leq n \\ \quad 0, & d \geq n+1 \end{cases} \\ |\mathrm{Im}(\beta_d)| &= \begin{cases} \binom{2g}{d}, & d \leq n \\ 0, & d \geq n+1 \end{cases} \end{aligned} \tag{46}$$

As a consequence of results from Section 1.3 we deduce the following proposition which is also used in the proof of Theorem 2.5.

Proposition 3.2. *Let $\alpha_p : H_p(F_n) \to H_p(SP^n(M_g))$ and $\beta_p : H_p(SP^n(M_{g,k})) \to H_p(SP^n(M_g))$ be the homomorphisms induced by the inclusions $F_n \hookrightarrow SP^n$ and $SP^n(M_{g,k}) \hookrightarrow SP^n(M_g)$. If $\alpha_p + \beta_p : H_p(F_n) \oplus H_p(SP^n(M_{g,k}) \to H_p(SP^n(M_g))$ is the map defined by $(\alpha_p + \beta_p)(x \oplus y) = \alpha_p(x) + \beta_p(y)$, then*

$$|\mathrm{Im}(\alpha_p) \cap \mathrm{Im}(\beta_p)| = \begin{cases} \binom{2g}{p}, & p \leq n-1 \\ 0, & p \geq n \end{cases} \tag{47}$$

Proof: By the results of Sections 1.3 and 1.4 the group $H_p(F_n)$ has the following decomposition

$$H_p(F_n) \cong \bigoplus_{\nu=0}^{+\infty} \bigoplus_{i+j=p} H_i(SP^\nu(M_g), SP^{\nu-1}(M_g)) \otimes H_j(\Delta(P_\nu)). \tag{48}$$

Define

$$H_p(F_n)^{(0)} := \bigoplus_{\nu=0}^{+\infty} H_p(SP^\nu(M_g), SP^{\nu-1}(M_g)) \otimes H_0(\Delta(P_\nu))$$

and let $\alpha_p' : H_p(F_n)^{(0)} \to H_p(SP^n(M_g))$ be the restriction of α_p on $H_p(F_n)^{(0)}$. Let us observe that

$$H_p(F_n)^{(0)} = \big(H_p(SP^{n-1}(M_g), SP^{n-2}(M_g)) \otimes \mathbb{Q}^k\big) \oplus H_p(SP^{n-2}(M_g)). \tag{49}$$

Moreover, $\mathrm{Im}(\alpha_p) = \mathrm{Im}(\alpha_p') = H_p(SP^{n-1}(M_g)) \subset H_p(SP^n(M_g))$.

By the analysis preceding Proposition 3.1 we know that $\mathrm{Im}(\beta_p)$ is spanned by classes $E_{(I,0)} = e_I = e_{i_1} \ldots e_{i_p}$ where $1 \leq i_1 < \ldots < i_p \leq 2g$ and $p \leq n$. A conclusion is that

$$\mathrm{Im}(\alpha_p) \cap \mathrm{Im}(\beta_p) = \mathrm{span}\{e_I \mid I : 1 \leq i_1 < \ldots < i_p \leq 2g,\, p \leq n-1\}$$

and a simple calculation completes the proof of the proposition. □

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